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Uncertainty relations and quadratic forms

F Gesztesy† and L Pittner
Institut für Theoretische Physik, Universität Graz, A-8010 Graz, Austria

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Abstract. We show that sesquilinear form methods can be used to extend the usual uncertainty relation for symmetric operators to a larger and more natural domain of states.

Consider symmetric operators A, B and the commutation relation

$$C = \mathrm{i}[A, B] \qquad \text{on} \quad D(C) = D(AB) \cap D(BA), \tag{1}$$

then one immediately obtains the inequality

$$\Delta_f A \,.\, \Delta_f B \geqslant \tfrac{1}{2}|(f, Cf)| \qquad \text{for all} \quad f \in D(C). \tag{2}$$

The symmetric operator C need not be essentially self-adjoint, but we assume that C allows self-adjoint extensions. Since the product $\Delta_f A \,.\, \Delta_f B$ is defined for all $f \in D(A) \cap D(B)$, it should be possible (at least under certain additional assumptions) to extend the inequality (2) to

$$\Delta_f A \,.\, \Delta_f B \geqslant \tfrac{1}{2}|(f, \tilde{C}f)| \qquad \text{for all} \quad f \in D(A) \cap D(B) \cap D(\tilde{C}), \tag{3}$$

where $\tilde{C}$ denotes a suitable self-adjoint extension of C.

In general, however, one may deduce from relation (1) only a weaker version of the inequality (3), where the expectation value $(f, \tilde{C}f)$ is replaced by some quadratic form (the weak commutator) (Kraus 1965). But in the special case where the closures $\bar{A}$, $\bar{B}$ and $\bar{C}$ are self-adjoint generators of a unitary representation of a two- or three-parameter Lie group, and therefore (1) is implied by the structure of the corresponding Lie algebra (Nelson 1959, Segal 1951), the uncertainty relation (2) can be extended to all states $f \in D(\bar{A}) \cap D(\bar{B}) \cap D(\bar{C})$ (Kraus 1967).

In this paper, by means of sesquilinear forms, we discuss an extension of the inequality (2) in those cases where the commutator C is semi-bounded. In particular we relate the possibility of extending (2) and (3) to properties of sesquilinear forms such as semi-boundedness and closability.

Our precise assumptions about A, B and C are:

(i) A and B are symmetric operators in a Hilbert space $\mathscr{H}$.

(ii) The commutator $C = \mathrm{i}[A, B]$ on $D(C) = D(AB) \cap D(BA)$ is densely defined in $\mathscr{H}$ and semi-bounded.

In the usual way we introduce

$$\Delta_f A = \|(A - \langle A \rangle_f)f\|, \qquad \langle A \rangle_f = (f, Af)/(f, f), \qquad f \in D(A),$$

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and obtain by means of Schwarz' inequality

$$
\begin{aligned}
&(\Delta_f A)^2 \,.\, (\Delta_f B)^2 \\
&\qquad \geqslant |((A-\langle A\rangle_f)f, (B-\langle B\rangle_f)f)|^2 \\
&\qquad = \tfrac{1}{4}[|(Af, Bf)-(Bf, Af)|^2 + |((A-\langle A\rangle_f)f, (B-\langle B\rangle_f)f) \\
&\qquad\quad + ((B-\langle B\rangle_f)f, (A-\langle A\rangle_f)f)|^2] \\
&\qquad \geqslant \tfrac{1}{4}|(Af, Bf)-(Bf, Af)|^2, \qquad f\in D(A)\cap D(B). \qquad (4)
\end{aligned}
$$

In order to relate the right-hand side of (4) to the commutator C, we introduce the sesquilinear forms

$$
\begin{aligned}
&q_0(f, g) = (f, Cg), && D(q_0) = D(C), \\
&q_1(f, g) = \mathrm{i}(Af, Bg) - \mathrm{i}(Bf, Ag), && D(q_1) = D(A)\cap D(B).
\end{aligned} \qquad (5)
$$

From (i) and (ii) we conclude that the form q_0 is densely defined, symmetric, semi-bounded and closable (Kato 1966). The self-adjoint operator C_0 associated with the closure $\bar{q}_0$ of q_0 is just the Friedrichs extension C_F of C:

$$
\bar{q}_0(f, g) = (f, C_0 g), \qquad f\in D(\bar{q}_0), \qquad g\in D(C_0); \qquad C_0 = C_F. \qquad (6)
$$

On the other hand, the form q_1, although densely defined and symmetric, need not be closable (cf example 1). If q_1 is semi-bounded and closable, then the self-adjoint operator C_1 associated with the closure $\bar{q}_1$ of q_1,

$$
\bar{q}_1(f, g) = (f, C_1 g), \qquad f\in D(\bar{q}_1), \qquad g\in D(C_1), \qquad (7)
$$

evidently is a self-adjoint extension of C, but *a priori* there is no connection between C_1 and the Friedrichs extension C_0 of C (cf example 2). Special assumptions which indeed imply $C_0 = C_1$ are:

(I) Suppose C is essentially self-adjoint; if q_1 is semibounded and closable then $C_0 = C_1$.

(II) Let B be bounded; if $D(C)$ is a core of A, then q_1 is semi-bounded and closable, and again $C_0 = C_1$.

Statement (I) only reflects the uniqueness of the self-adjoint extension of an essentially self-adjoint operator, while for the proof of (II) it suffices to note that here q_1 is a restriction of $\bar{q}_0$. Thus we must distinguish whether the form q_1 is semi-bounded and closable (case 1) or not (case 2).

Case 1. From (4) and (7) we immediately obtain the extended inequality (3) in the form

$$
\Delta_f A \,.\, \Delta_f B \geqslant \tfrac{1}{2}|(f, C_1 f)|, \qquad f\in D(A)\cap D(B)\cap D(C_1). \qquad (8)
$$

Case 2. Here one only gets the weaker relation (Kraus 1965)

$$
\Delta_f A \,.\, \Delta_f B \geqslant \tfrac{1}{2}|q_1(f, f)|, \qquad f\in D(A)\cap D(B). \qquad (9)
$$

In order to illustrate these statements we turn to explicit examples. We have already noted that although q_0 is closable by definition via C, the form q_1 need not be closable; actually the pair {angle, angular momentum} represents a prominent example for such a situation.

Example 1. In the Hilbert space $\mathscr{H}=L^2([-\pi,+\pi];\mathrm{d}\phi)$ define the self-adjoint operators

$$L=\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\phi}\quad\text{on } D(L)=\{f|f\in A([-\pi,+\pi]);f,f'\in\mathscr{H};f(-\pi)=f(+\pi)\},$$
$$\Phi=\phi\qquad\text{on } D(\Phi)=\mathscr{H},\tag{10}$$

where $A([\alpha,\beta])$ denotes the set of functions which are absolutely continuous on $[\alpha,\beta]$. Then one immediately obtains

$$\mathrm{i}(Lf,\Phi g)-\mathrm{i}(\Phi f,Lg)=(f,g)-2\pi\overline{f(\pi)}g(\pi),\qquad f,g\in D(L).\tag{11}$$

L is not essentially self-adjoint on

$$D([L,\Phi])=\{f|f\in D(L);f(-\pi)=f(+\pi)=0\},\tag{12}$$

and the spectra are

$$\Sigma(L)=\mathscr{Z},\qquad \Sigma(\Phi)=[-\pi,+\pi].\tag{13}$$

We identify

$$C=\mathrm{i}[L,\Phi]=1\qquad\text{on}\quad D(C)=D([L,\Phi])\tag{14}$$

and

$$q_1(f,g)=(f,g)-2\pi\overline{f(\pi)}g(\pi)\qquad\text{on}\quad D(q_1)=D(L).\tag{15}$$

The operator C is essentially self-adjoint, but $\overline{f(\pi)}g(\pi)$ (and hence q_1) is the canonical example of a non-closable symmetric form. Thus instead of the inequality (8) we only have (Kraus 1965)

$$\Delta_fL\,.\,\Delta_f\Phi\geqslant\tfrac{1}{2}|\,\|f\|^2-2\pi|f(\pi)|^2|,\qquad f\in D(L).\tag{16}$$

For further discussions of this example compare Carruthers and Nieto (1968), Judge (1964), Kraus (1965), and Lévy-Leblond (1976). Concerning the pair {phase operator, number operator} (Garrison and Wong 1970), exactly the same conclusions can be drawn. More generally this situation occurs in all cases where C is essentially self-adjoint and $C\geqslant\gamma>0$ (or $C\leqslant\gamma<0$), while A or B has an eigenstate which lies in $D(A)\cap D(B)\cap D(\bar{C})$. Suppose C essentially self-adjoint and $C\geqslant\gamma>0$, $Af_\alpha=\alpha f_\alpha$ with $f_\alpha\in D(A)\cap D(B)\cap D(\bar{C})$; if q_1 were semi-bounded and closable, then one would conclude

$$\Delta_fA\,.\,\Delta_fB\geqslant\tfrac{1}{2}|(f,\bar{C}f)|>0\qquad\text{for all}\quad f\in D(A)\cap D(B)\cap D(\bar{C}),$$

in contradiction to $\Delta_{f_\alpha}A=0$.

Next we consider a special case where q_1 is semi-bounded and closable, but the self-adjoint operator C_1 associated with $\bar{q}_1$ does not coincide with C_0.

Example 2. In the Hilbert space $\mathscr{H}=L^2([0,1];\mathrm{d}x)$ consider the symmetric operators

$$A=\frac{\mathrm{i}}{3}\frac{\mathrm{d}^3}{\mathrm{d}x^3}\quad\text{on } D(A)=\{f|f,f',f''\in A([0,1]);f,f',f'',f'''\in\mathscr{H};$$
$$f'(0)=f'(1)=f''(0)=f''(1)=0\},\tag{17}$$
$$B=x\quad\text{on } D(B)=\mathscr{H}.$$

Then obviously

$$\mathrm{i}(Af, Bg) - \mathrm{i}(Bf, Ag) = (f', g'), \qquad f, g \in D(A), \tag{18}$$

furthermore

$$C = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \quad \text{on } D(C) = \{f \mid f, f', f'' \in A([0,1]); f, f', f'', f''' \in \mathcal{H};$$
$$f(0) = f(1) = f'(0) = f'(1) = f''(0) = f''(1) = 0\}, \tag{19}$$

and therefore

$$q_0(f, g) = (f', g'), \qquad f, g \in D(q_0) = D(C), \tag{20}$$

whereas

$$q_1(f, g) = (f', g'), \qquad f, g \in D(q_1) = D(A). \tag{21}$$

The corresponding closures are

$$\bar{q}_0(f, g) = (f', g'), \qquad D(\bar{q}_0) = \{f \mid f \in A([0,1]); f, f' \in \mathcal{H}; f(0) = f(1) = 0\}, \tag{22}$$

$$C_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad D(C_0) = \{f \mid f, f' \in A([0,1]); f, f', f'' \in \mathcal{H}; f(0) = f(1) = 0\}, \tag{23}$$

and

$$\bar{q}_1(f, g) = (f', g'), \qquad D(\bar{q}_1) = \{f \mid f \in A([0,1]); f, f' \in \mathcal{H}\}, \tag{24}$$

$$C_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad D(C_1) = \{f \mid f, f' \in A([0,1]); f, f', f'' \in \mathcal{H}; f'(0) = f'(1) = 0\}. \tag{25}$$

Here of course we can write

$$\Delta_f A \,.\, \Delta_f B \geqslant \tfrac{1}{2}(f, C_1 f) \qquad \text{for all} \quad f \in D(A), \tag{26}$$

but from (23) and (25) we see that C_1 is not identical with C_0; actually we have $C_1 \geqslant 0$, whereas $C_0 \geqslant \pi^2$.

After these two examples, which illustrate some of the complications one may encounter in trying to generalise (2)–(8), we now consider the Schrödinger representation of the canonical commutation relation for one degree of freedom.

Example 3. In the Hilbert space $\mathcal{H} = L^2(\mathcal{R}; \mathrm{d}x)$, consider the Schrödinger couple

$$P = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}, \qquad D(P) = \{f \mid f \in A_{\mathrm{loc}}(\mathcal{R}); f, f' \in \mathcal{H}\},$$
$$Q = x, \qquad D(Q) = \{f \mid f, xf \in \mathcal{H}\}, \tag{27}$$

where $A_{\mathrm{loc}}(\mathcal{R})$ denotes the set of locally absolutely continuous functions on $\mathcal{R}$. Here we have

$$q_1(f, g) = \mathrm{i}(Pf, Qg) - \mathrm{i}(Qf, Pg) = (f, g), \qquad D(q_1) = D(P) \cap D(Q),$$
$$\bar{q}_1(f, g) = (f, g), \qquad D(\bar{q}_1) = \mathcal{H}. \tag{28}$$

On the other hand,

$$C = \mathrm{i}[P, Q] = 1 \qquad \text{on} \quad D(C) = D([P, Q]). \tag{29}$$

Applying (I) we get $C_0 = C_1 = 1$ on $\mathscr{H}$. The Heisenberg uncertainty relation thus reads

$$\Delta_f P \,.\, \Delta_f Q \geq \tfrac{1}{2}(f, f) \qquad \text{for all} \quad f \in D(P) \cap D(Q). \tag{30}$$

Since 1, P and Q are the generators of the so-called Heisenberg group with Weyl's relation

$$\mathrm{e}^{\mathrm{i}sP}\, \mathrm{e}^{\mathrm{i}tQ} = \mathrm{e}^{\mathrm{i}st}\, \mathrm{e}^{\mathrm{i}tQ}\, \mathrm{e}^{\mathrm{i}sP}, \qquad s, t \in \mathscr{R}, \tag{31}$$

as group multiplication law, inequality (30) is an example covered by the methods developed for representations of Lie groups in Hilbert spaces (Kraus 1967).

If some pair $\{A, B\}$ of self-adjoint operators in a Hilbert space $\mathscr{H}$ satisfies Weyl's form of the canonical commutation relation (Putnam 1967)

$$\mathrm{e}^{\mathrm{i}sA}\, \mathrm{e}^{\mathrm{i}tB} = \mathrm{e}^{\mathrm{i}st}\, \mathrm{e}^{\mathrm{i}tB}\, \mathrm{e}^{\mathrm{i}sA}, \qquad s, t \in \mathscr{R}, \tag{32}$$

then necessarily

$$\mathrm{i}(Af, Bg) - \mathrm{i}(Bf, Ag) = (f, g), \qquad f, g \in D(A) \cap D(B), \tag{33}$$

and therefore the inequality

$$\Delta_f A \,.\, \Delta_f B \geq \tfrac{1}{2}(f, f), \qquad f \in D(A) \cap D(B), \tag{34}$$

holds. Although Weyl's relation (32) implies (33) and (34), the converse conclusion cannot be drawn in general; there exist examples where the uncertainty relation (34) is fulfilled but Weyl's relation (32) does not hold. (But one can prove (Tillmann 1963) that equation (33) supplemented by the condition $(A + \mathrm{i}B)^* = \overline{(A - \mathrm{i}B)}$ suffices to guarantee Weyl's relation (32).) We finally present such an example (Fuglede 1967).

Example 4. In the Hilbert space $\mathscr{H} = L^2(\mathscr{R}; \mathrm{d}x)$, define the essentially self-adjoint operators

$$\dot{A} = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x} + \exp(\sqrt{2\pi}x), \qquad \dot{B} = x + \exp\left(-\sqrt{2\pi} \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x}\right),$$

$$D(\dot{A}) = D(\dot{B}) = \text{linear span}\, \{x^n \exp(-rx^2 + cx) | n \in \mathscr{N}, r > 0, c \in \mathscr{C}\}, \tag{35}$$

and denote their closures by A and B respectively. Then one obtains (Fuglede 1967)

$$q_1(f, g) = \mathrm{i}(Af, Bg) - \mathrm{i}(Bf, Ag) = (f, g) \qquad \text{on} \quad D(A) \cap D(B),$$

$$C = 1 \qquad \text{on} \quad D(AB) \cap D(BA), \tag{36}$$

which implies, just as in the foregoing example,

$$C_0 = C_1 = 1 \qquad \text{on} \quad \mathscr{H} \tag{37}$$

and therefore

$$\Delta_f A \,.\, \Delta_f B \geq \tfrac{1}{2}(f, f) \qquad \text{on} \quad D(A) \cap D(B). \tag{38}$$

But this pair $\{A, B\}$ is not unitarily equivalent to any direct sum of Schrödinger couples $\{P, Q\}$, i.e. A, B do not obey Weyl's relation (32).

References

Carruthers P and Nieto M M 1968 *Rev. Mod. Phys.* **40** 411–40

Fuglede B 1967 *Math. Scand.* **20** 79–88
Garrison J C and Wong J 1970 *J. Math. Phys.* **11** 2242–9
Judge D 1964 *Nuovo Cim.* **31** 332–40
Kato T 1966 *Perturbation Theory for Linear Operators* (New York: Springer)
Kraus K 1965 *Z. Phys.* **188** 374–7
—— 1967 *Z. Phys.* **201** 134–41
Lévy-Leblond J M 1976 *Ann. Phys., NY* **101** 319–41
Nelson E 1959 *Ann. Math.* **70** 572–615
Putnam C R 1967 *Commutation Properties of Hilbert Space Operators and Related Topics* (New York: Springer)
Segal I E 1951 *Duke Math. J.* **18** 221–65
Tillmann H G 1963 *Acta Sci. Math., Szeged* **24** 258–70